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The Korteweg–de Vries equation and water waves. Solutions of the equation. Part 1

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The method of solution of the Korteweg–de Vries equation outlined by Gardner *et al.* (1967) is exploited to solve the equation. A convergent series representation of the solution is obtained, and previously known aspects of the solution are related to this general form. Asymptotic properties of the solution, valid for large time, are examined. Several simple methods of obtaining approximate asymptotic results are considered.

1. Introduction and main conclusions

The Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

describes the nonlinear propagation of waves in many dispersive media where dissipation is absent. The ingenious work of Gardner *et al.* (1967) reduces the problem of solving (1) to that of solving the linear integral equation of Gelfand & Levitan (1955), in a form due to Marchenko (cf. Agranovich & Marchenko 1963),

$$K(x,y) + B(x+y) + \int_x^\infty K(x,z)\,B(z+y)\,dz = 0 \quad (y > x), \tag{2}$$

where $B(x+y)$ depends on the initial data $u(x,0)$. The theory imposes two limitations on the initial data. First, to guarantee the existence of a function $u(x,t)$ for which u_{xxx} is defined almost everywhere for $0 \leqslant t < \infty$, it is necessary that the initial data satisfy

$$\int_{-\infty}^{\infty} [u^2 + (u_x)^2 + (u_{xx})^2 + (u_{xxx})^2]\,dx < \infty \tag{3}$$

(Bona & Smith 1973). Second, to guarantee the solvability of (2) for $0 \leqslant t < \infty$, $u(x,0)$ must be locally integrable and must satisfy

$$\int_{-\infty}^{\infty} (1+|x|)\,|u|\,dx < \infty \tag{4}$$

(Faddeev 1958). The two conditions, as well as their respective purposes, are quite independent of each other. We shall assume that both are satisfied throughout the present discussion.

Determination of the exact solution of (1), for acceptable initial data, is still a substantial task. For large time, however, the solution acquires a comparatively simple structure which one can approximate to any desired accuracy. Aspects of this asymptotic solution have been discussed by several other authors, whom we mention in context.

In part 1 of the present paper we examine the properties of the solution of the Korteweg–de Vries equation that evolves from arbitrary initial data satisfying (3) and (4). We examine the exact solution in some detail in § 2. In § 3 we consider several approximate methods to obtain information about the asymptotic solution as $t \to \infty$. In § 4 we relate the solution obtained by means of (2) to the solution obtained by iteration directly in the differential equation. In part 2 we shall compare these results with data from experiments on water waves, the field in which the equation was originally derived (Korteweg & de Vries 1895).

Among the features of the asymptotic solution, the following appear to be the most important.

(i) *An arbitrary, initial disturbance* evolves into a finite number of permanent waves, called solitons, and an oscillatory wave train, which disperses.

(ii) *Cnoidal waves*, the uniform wave train solutions of (1), play no part whatever in the evolution of disturbances that satisfy (4).

(iii) *The solitons.* (*a*) The solitons are positive ($u(x,t) \geqslant 0$), for (1) as written. Each soliton travels with a positive speed which is proportional to its amplitude. Thus, the solitons are eventually ordered by amplitude. No acceptable initial data can produce two solitons with the same velocity. (*b*) The net effect on a soliton from having experienced an interaction is a phase shift. For two solitons, the interaction advances the faster wave and retards the slower one. (*c*) If

$$\int_{-\infty}^{\infty} u(x,0)\,dx > 0,$$

at least one soliton emerges. If $u(x,0) \leqslant 0$, no solitons emerge. (*d*) The number of solitons N that emerge from the initial data depends on the parameter $(\bar{u}l^2)$, where $\bar{u}$ is a typical amplitude of the initial data, and l is a typical length. For initial data of finite extent (such as the data of the experiments in part 2),

if $$\tan(l_2 Q_2^{\frac{1}{2}}) \leqslant \frac{2Q_2^{\frac{1}{2}}}{Q_2 - Q_3}, \quad \frac{l_2 Q_2^{\frac{1}{2}}}{\pi} - 1 < N < \frac{l_1 Q_1^{\frac{1}{2}}}{\pi} + 1; \tag{5a}$$

if $$\tan(l_2 Q_2^{\frac{1}{2}}) \geqslant \frac{2Q_2^{\frac{1}{2}}}{Q_2 - Q_3}, \quad \frac{l_2 Q_2^{\frac{1}{2}}}{\pi} \leqslant N < \frac{l_1 Q_1^{\frac{1}{2}}}{\pi} + 1, \tag{5b}$$

where l_1, l_2, Q_1, Q_2 and Q_3 are shown in figure 1. For arbitrary initial data,

$$N \leqslant 1 + \int_{-\infty}^{\infty} |x|\, q(x)\,dx, \tag{6a}$$

where $$q(x) = \begin{cases} u(x,0) & \text{if} \quad u(x,0) \geqslant 0, \\ 0 & \text{if} \quad u(x,0) < 0. \end{cases} \tag{6b}$$

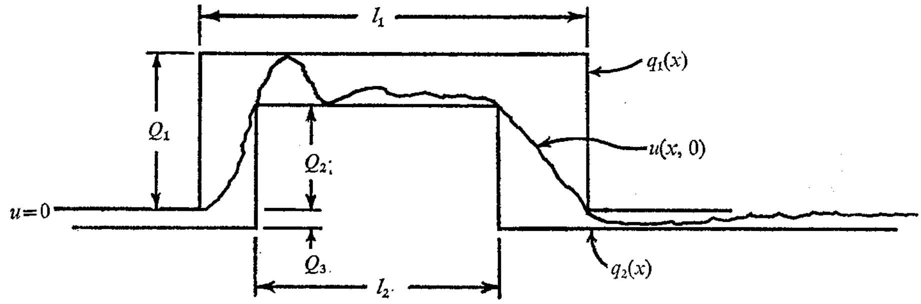

FIGURE 1. Bounds for the initial data.

A third bound has been found by Karpman & Sokolov (1968), the validity of which requires that $u(x,0) \geqslant 0$ and that N be large. (*e*) The amplitude of the solitons can be estimated by a Rayleigh–Ritz procedure. The largest amplitude is $2|\lambda|$, where

$$\lambda = \min_{\psi(x)} \left\{ \frac{\int_{-\infty}^{\infty} \left\{ \left(\frac{d\psi}{dx}\right)^2 - u\psi^2 \right\} dx}{\int_{-\infty}^{\infty} \psi^2(x) dx} \right\}. \tag{7}$$

The amplitude does not exceed $2u_0$, where $u_0 = \max_x \{u(x,0)\}$. It approaches $2u_0$ as the size of the initial disturbance (and the number of nascent solitons) increases. For a small disturbance with

$$\int_{-\infty}^{\infty} u(x,0)\, dx \geqslant 0,$$

$$|\lambda| \doteq \left[\frac{1}{2}\int_{-\infty}^{\infty} u(x,0)\, dx\right]^2. \tag{8}$$

(iv) *The oscillatory wave train.* (*a*) The group velocity of the oscillatory wave train, or 'tail', is non-positive. (*b*) The tail has a dispersive character which is similar to that of the solution of the corresponding linear equation. Specifically, short waves dominate near the rear of the tail, longer waves move to the front. The amplitudes of individual wave crests decay as $t^{-\frac{1}{3}}$, which suggests that the decay rate of the entire tail is algebraic. (*c*) The naive perturbation expansion of the solution of (1), in powers of its amplitude, can be made to converge to the solution if and only if no solitons exist. When it converges, this expansion coincides (term-by-term) with that obtained by solving (2). (*d*) The similarity solution obtained by Berezin & Karpman (1964) emanates from initial data that violate (4).

(v) *The general solution.* An initial disturbance, in general, evolves into both solitons and a tail. These separate as $t \to \infty$, because of their respective speeds. There appears to be no permanent effect on the solitons from the interaction with the tail, or *vice versa*. The final shape, speed and phase of each soliton apparently are unaffected by the tail, and (for the special case of only one soliton) the asymptotic behaviour of the tail is unaffected by the soliton.

2. The exact solution

2.1. *The method of solution*

The remarkable method of solution of (1), as outlined by Gardner *et al.*, follows.

(i) Solve the ordinary scattering problem, using the initial data $u(x,0)$ as the potential; i.e. solve the eigenvalue problem

$$\frac{d^2}{dx^2}\psi(x)+(\lambda+u(x,0))\,\psi(x) = 0 \tag{9}$$

for both positive and negative λ. For $\lambda \geqslant 0$ ($\lambda = k^2$), the boundary conditions for $\psi(x)$ are

$$\left.\begin{aligned}\psi(x) &\sim \exp\{-ikx\}+b(k)\exp\{ikx\} \quad (x\to\infty),\\ \psi(x) &\sim a(k)\exp\{-ikx\} \quad (x\to-\infty).\end{aligned}\right\} \tag{10}$$

One can show that $|a|^2+|b|^2 = 1$, from which it follows that

$$|b(k)| < 1 \quad \text{if} \quad k \neq 0. \tag{11}$$

For $\lambda < 0$, (9) has a bounded solution only for a discrete set of eigenvalues ($\lambda_n = -K_n^2$). If $u(x,0)$ satisfies (4), the number N of such eigenvalues is finite. The corresponding eigenfunctions $\psi_n(x)$ can each be normalized such that

$$\int_{-\infty}^{\infty}\psi_n^2(x)\,dx = 1.$$

After normalizing each eigenfunction $\psi_n(x)$, find its asymptotic behaviour as $x\to+\infty$:

$$\psi_n(x) \sim c_n\exp\{-K_n x\}. \tag{12}$$

(ii) Construct the function $B(r,t)$ and $a(k,t)$:

$$a(k,t) = a(k,0),$$

$$B(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\exp\{8ik^3t+ikr\}\,dk+\sum_{n=1}^{N}c_n^2\exp\{8K_n^3t-K_nr\}. \tag{13}$$

The function B now contains all the information given by $u(x,0)$. As one might expect, the precise determination of B is rather tedious unless $u(x,0)$ is quite simple. In what follows, we shall find it convenient to determine $u(x,t)$ in terms of B, rather than $u(x,0)$, and to suppress time dependence:

$$B(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{b}(k)\exp\{ikr\}\,dk+\sum_{n=1}^{N}\tilde{c}_n^2\exp\{-K_nr\}, \tag{13a}$$

where $\qquad \tilde{b}(k) = b(k)\exp\{8ik^3t\}, \quad \tilde{c}_n = c_n\exp\{4K_n^3t\}.$

(iii) Solve the Gelfand–Levitan equation (2) for $K(x,y;t)$. Note that both x and t enter this equation merely as parameters. Then

$$u(x,t) = 2\frac{d}{dx}K(x,x;t). \tag{14}$$

This is the remarkable result of Gardner *et al.* (1967).

2.2. *The discrete spectrum*

The method of solution employed here is essentially that outlined by Gelfand & Levitan in their original work. The structure of the general solution is somewhat clearer if we first restrict our attention to cases in which B depends only on the discrete spectrum:

$$B_D(r,t) = \sum_{n=1}^{N} c_n^2 \exp\{8K_n^3 t - K_n r\}. \tag{15}$$

The integral equation (2) then has a degenerate kernel, and may be solved in closed form:

$$K_D(x,y;t) = -\sum_{n,m=1}^{N} \tilde{c}_n \exp\{-K_n x\} \times \left(\delta_{mn} + \frac{\tilde{c}_n \tilde{c}_m \exp\{-(K_n+K_m)x\}}{K_n+K_m}\right)^{-1} \tilde{c}_m \exp\{-K_m y\},$$

where δ_{mn} is the Kronecker delta. The solution of (1) may be written as

$$u(x,t) = 2\frac{d^2}{dx^2}\log(\det(Pmn)), \tag{16}$$

where
$$Pmn = \Sigma\delta_{mn} + \frac{\tilde{c}_m \tilde{c}_n \exp\{-(K_m+K_n)x\}}{K_m+K_n},$$

a result first obtained by Kay & Moses (1956). The discrete spectrum yields solitons, as noted by Gardner *et al.* If $N = 1$, $u(x,t)$ reduces to a solitary wave:

$$u(x,t) = 2K_1^2 \operatorname{sech}^2\{K_1(x - x_1 - 4K_1^2 t)\}, \tag{17}$$

where
$$K_1 x_1 = \log(c_1/(2K_1)^{\frac{1}{2}}).$$

Thus, the eigenvalue $(-K_1^2)$ determines the amplitude of the wave and the coefficient (c_1) determines its phase. The case $N = 2$ provides a simple example of how solitons interact:

$$u(x,t) = \left[8K_1^2\left\{\frac{1}{f_2} + \left(\frac{K_1-K_2}{K_1+K_2}\right)f_2\right\}^2 + 8K_2^2\left\{\frac{1}{f_1} + \left(\frac{K_2-K_1}{K_2+K_1}\right)f_1\right\}^2\right] \times \left[\left\{\frac{1}{f_1 f_2} + \frac{f_1}{f_2} + \frac{f_2}{f_1} + \left(\frac{K_1-K_2}{K_1+K_2}\right)^2 f_1 f_2\right\}^2\right]^{-1}, \tag{18}$$

where
$$f_1 = \tilde{c}_1 \exp\{-K_1 x\}/(2K_1)^{\frac{1}{2}}, \quad f_2 = \tilde{c}_2 \exp\{-K_2 x\}/(2K_2)^{\frac{1}{2}}.$$

As $t \to \pm\infty$ (for $K_1 > K_2 > 0$),

$$u(x,t) \to 2K_1^2 \operatorname{sech}^2\{K_1(x - x_1^{\pm} - 4K_1^2 t)\} + 2K_2^2 \operatorname{sech}^2\{K_2(x - x_2^{\pm} - 4K_2^2 t)\},$$

$$x_1^- = \frac{1}{2K_1}\ln\frac{c_1^2}{2K_1} - \frac{1}{K_1}\ln\left|\frac{K_1+K_2}{K_1-K_2}\right|, \quad x_2^- = \frac{1}{2K_2}\ln\frac{c_2^2}{2K_2},$$

$$x_1^+ = x_1^- + \frac{1}{K_1}\ln\left|\frac{K_1+K_2}{K_1-K_2}\right|, \quad x_2^+ = x_2^- - \frac{1}{K_2}\ln\left|\frac{K_1+K_2}{K_1-K_2}\right|.$$

Each soliton retains its identity despite the interaction, experiencing at most a phase shift, a result previously obtained by Lax (1968). Equation (18) is plotted

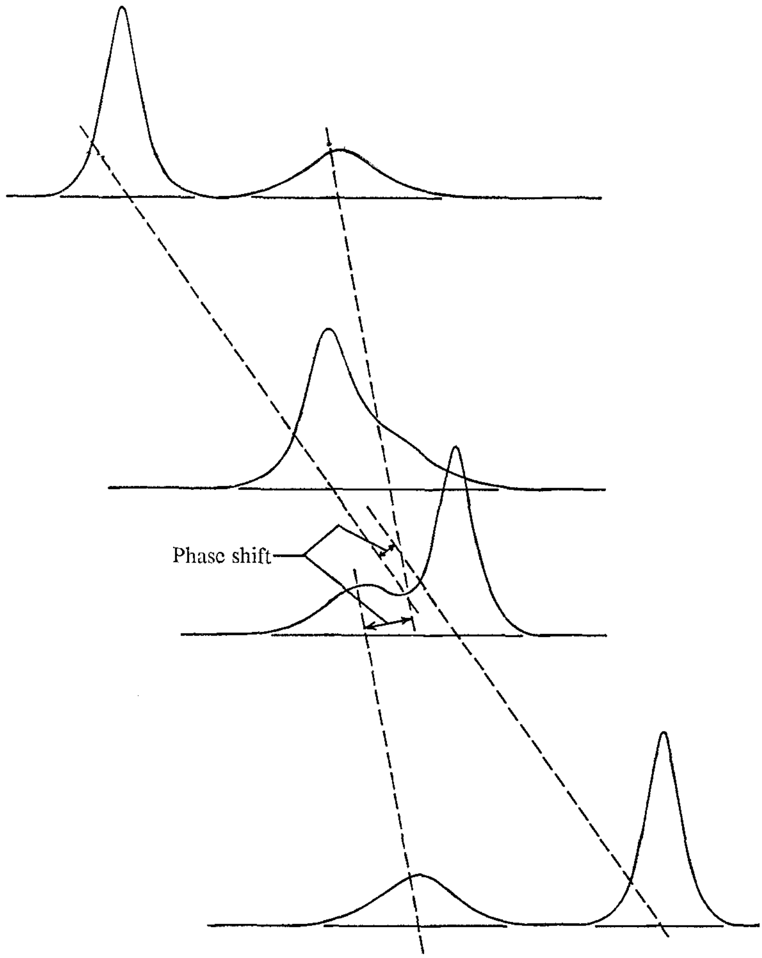

FIGURE 2. Interaction of two solitons at succeeding times. $K_1 = 1$, $K_2 = 2$.

in figure 2 for typical values of the parameters. Finally, it should be noted from (18) that $K_1^2 = K_2^2$ produces one, not two solitons. Hence, two solitons of identical size, travelling in the same direction, are impossible. This result was suggested by Benney & Luke (1964) and Byatt-Smith (1971).

2.3. *The continuous spectrum*

Next, consider a purely continuous spectrum:

$$B_c(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\exp\{8ik^3t + ikr\}\,dk. \tag{19}$$

Equation (2) is still solvable, not because the kernel is degenerate, but because it is small. More precisely, if $\|\mathring{B}_c\|$ denotes the L_2-norm of the integral operator whose kernel is B_c, then we shall see that $\|\mathring{B}_c\| < 1$ and that the homogeneous integral equation

$$\phi(y) + \int_x^{\infty} \phi(z)\,B_c(z+y)\,dz = 0 \quad (y > x) \tag{20}$$

has only the trivial solution. Consider the real functions $\phi(y)$ which vanish identically for $y < x$, and for which

$$\|\phi\|^2 = \int_{-\infty}^{\infty} \phi(y)\,\phi(y)\,dy = \int_{x}^{\infty} \phi(y)\,\phi(y)\,dy < \infty.$$

Then
$$\|\phi\|^2 + \iint_{-\infty}^{\infty} \phi(z)\,B_c(z+y)\,\phi(y)\,dz\,dy = 0.$$

Using (19) and the definition of a Fourier transform,

$$\hat{\phi}(k) = \int_{-\infty}^{\infty} \phi(y)\exp\{-iky\}\,dy,$$

and reversing the order of integration, we obtain

$$\|\phi\|^2 + \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{b}(k)\,\hat{\phi}^2(-k)\,dk = 0.$$

If $\hat{\phi}(k) \neq 0$ for some $k \neq 0$, then

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{b}(k)\,\hat{\phi}^2(-k)\,dk\right| \leqslant \max_{k\neq 0}\{|\tilde{b}(k)|\}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\phi}(k)|^2\,dk < \|\phi\|^2,$$

by Parseval's relation and (11). This method fails if $\hat{\phi}(k) = c\delta(k)$, but no $\phi(y)$ in the class considered can have such a transform. Thus we find both that (20) has only the trivial solution, and that $\|\mathring{B}_c\| < 1$ on the class of functions considered. It follows that the solution of (2) may be represented as a convergent Neumann series:

$$K_c(x,y;t) = -B_c(x+y;t) + \int_x^{\infty} B_c(x+z;t)\,B_c(z+y;t)\,dz - \iint_x^{\infty} B_c(x+z_1;t) \times B_c(z_1+z_2;t)\,B_c(z_2+y;t)\,dz_1\,dz_2 + \ldots, \quad (21)$$

where B_c is given by (19). The solution of (1), corresponding to (21), is given by

$$u(x) = -2\left[\frac{\partial}{\partial x}B(2x) + 2\iint_x^{\infty} B_x B^2 (dz)^2 + 2\iiiint_x^{\infty} B_x B^4 (dz)^4 + \ldots\right] + 4K(x,x)\left[B(2x) + \iint_x^{\infty} B^3 (dz)^2 + \iiiint_x^{\infty} B^5 (dz)^4 + \ldots\right], \quad (22)$$

where time dependence has been suppressed. Alternatively,

$$u(x) = -2\frac{\partial}{\partial x}B(2x) + 4B^2(2x) + 4\left[B(2x)\int_x^{\infty} B^2\,dz - \iint_x^{\infty} B_x B^2 (dz)^2\right] + \ldots$$

expresses the solution in powers of B. With sufficient perseverance, one can verify directly that (22) satisfies the differential equation (1), to any desired order (in powers of B).

The question of existence and uniqueness, for the purely continuous spectrum, can be answered as follows. Faddeev (1958) showed that the requirements on $u(x, 0)$ of continuity and (4) are equivalent to the following conditions:

$$\left.\begin{aligned} |b(k)| &< 1, \quad \text{if } k \neq 0 \quad \text{and} \quad k \text{ real}, \\ |b(k)| &= O(|k|^{-1}) \quad \text{as} \quad k \to \pm\infty, \\ a(k) &= 1 + O(|k|^{-1}), \quad \operatorname{Im} k \geqslant 0, \ |k| \to \infty. \end{aligned}\right\} \tag{23}$$

For $t > 0$, $\tilde{b}(k)$ and $a(k)$ evolve as shown in (13). Substituting, we see that (23) is satisfied uniformly in time, so that $u(x, t)$ satisfies (4) for $0 \leqslant t < \infty$. The infinite series (22), which converges (in the mean) for $0 \leqslant t < \infty$, formally satisfies the differential equation and therefore represents the unique solution of (1) that evolves from the given initial condition.

Consider next the asymptotic behaviour of the solution as $t \to \infty$. In the linearized form of (1), without the term $6uu_x$, the fundamental solution of the equation is an Airy function. It follows that any smooth, absolutely integrable initial condition evolves into a slowly varying wave train, oscillating about $u = 0$. Its group velocity is such that for any large t, wavenumber k dominates at the location given by

$$x/t = -3k^2, \tag{24}$$

and the maximum amplitude of the wave train decays as $t^{-\frac{1}{3}}$.

In the nonlinear problem, all of this information is more elusive. We show in § 4 that if the spectrum is purely continuous, the asymptotic solution is still a slowly varying wave train, whose group velocity is still described by (24). We next show that the amplitudes of individual wave crests decay as $t^{-\frac{1}{3}}$, when no solitons exist.

Divide (1) by u and obtain

$$(\ln |u|)_t + 6u_x + \frac{u_{xx}u_x}{u^2} + \left(\frac{u_{xx}}{u}\right)_x = 0. \tag{25}$$

Let $X(t)$ denote the position of a local maximum of $|u|$ (i.e. a wave crest or trough), and let

$$M(t) = \ln |u| \quad \text{at} \quad x = X(t). \tag{26}$$

Then

$$\frac{dM}{dt} = \frac{1}{u}\frac{\partial u}{\partial t},$$

because $u_x = 0$ at a crest. Substituting into (25),

$$\frac{dM}{dt} = \left(-\frac{u_{xx}}{u}\right)_x. \tag{27}$$

$-u_{xx}/u$ is positive near the crests, and is a measure of the local curvature. Specifically, for a fixed, large t, $u(x, t)$ has a Fourier transform:

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k, t) \exp\{ikx\}\, dk.$$

If $u_{xx}(x, t)$ is sufficiently well-behaved,

$$u_{xx}(x, t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} k^2\hat{u}(k, t) \exp\{ikx\}\, dk.$$

Evaluate both integrals by stationary phase at the same x and (large) t:

$$u_{xx}/u \doteq k^2, \tag{28}$$

where k is the dominant wavenumber at that particular (x,t). As shown in § 4, the appropriate wavenumber is given by (24). Moreover, for fixed large t, the spatial variation of k^2 is also given by (24). Substituting (24) and (28) in (27),

$$\frac{dM}{dt} = -\frac{1}{3t}.$$

Thus, if one follows an individual wave crest (at its phase speed), one observes

$$u = O(t^{-\frac{1}{3}}) \quad \text{as} \quad t \to \infty. \tag{29}$$

The most important decay rate is that of the envelope of all the waves, and this need not decay as $t^{-\frac{1}{3}}$, since new crests can appear. However, since every crest within the envelope decays as $t^{-\frac{1}{3}}$, it suggests that the decay rate is algebraic (i.e. t^{-p}, $p > 0$) when no solitons exist.

Berezin & Karpman (1964) obtained a similarity solution to (1) of the form

$$u(x,t) = t^{-\frac{2}{3}}f(\bar{x}) \quad (\bar{x} = x/t^{\frac{1}{3}}). \tag{30}$$

The same authors (1967) also found the initial conditions that yield (30):

$$u(x,0) = c\delta'(x).$$

Thus, (30) evolves from initial data that violate (3) and lies beyond the range of the present analysis. Karpman (1967) also found a more general solution of (1), which suggests that $t^{-\frac{2}{3}}$ cannot be uniformly valid for a bounded solution.

2.4. *The complete spectrum*

The solution of (1) that evolves from a purely discrete or continuous spectrum consists of solitons or an oscillatory tail, respectively. When the spectrum is mixed, the solution contains both these parts, plus some interaction terms. The method of solution is a composite of the two methods used above. Schematically, (2) becomes

$$(I+\mathring{B}_D)K + \mathring{B}_c K = -(B_D + B_c),$$

where we again use $(\mathring{B})$ to denote an integral operator whose kernel is the function B. Thus,

$$(I+(I+\mathring{B}_D)^{-1}\mathring{B}_c)K = -(I+\mathring{B}_D)^{-1}(B_D+B_c). \tag{31}$$

We must show that $\|(I+\mathring{B}_D)^{-1}\mathring{B}_c\| \leqslant \|\mathring{B}_c\| < 1$, to show that this equation has a unique solution. We need only show that

$$\|(I+\mathring{B}_D)\phi\| \geqslant \|\phi\| \quad \text{for all } \phi,$$

which is a straightforward computation:

$$\begin{aligned}\|(I+\mathring{B}_D)\phi\|^2 &= \int_x^\infty \left|\phi(y) + \sum_n c_n^2 \exp\{-K_n y\}\int_x^\infty \exp\{-K_n z\}\phi(z)\,dz\right|^2 dy \\ &= \|\phi\|^2 + 2\sum_n \left\{c_n^2|\phi_n|^2 + \int_x^\infty \left|\sum_n c_n^2 \exp\{-K_n y\}\phi_n\right|^2 dy\right\} \\ &\geqslant \|\phi\|^2,\end{aligned}$$

where $$\phi_n = \int_x^\infty \exp\{-K_n z\}\,\phi(z)\,dz.$$

A Neumann series therefore exists for this problem as well.

The general series solution for an arbitrary mixed spectrum is rather cumbersome, and is omitted. In any particular problem, the solution is obtained by following the procedure suggested by (31): first, invert $(I+\mathring{B}_D)$ as in § 2.2, then solve the new integral equation (31) by iteration as in § 2.3. As $t \to \infty$, this solution of the general problem acquires a rather simple form.

As an example, the case $N = 1$ displays the interaction between a soliton and the oscillatory tail. Retaining the first four orders of iteration and suppressing time dependence, we find

$$K(x,x) = K_c(x,x) - c_a^2 \exp\{-2k_1 x\}\left[1+\frac{c_b^2\exp\{-2k_1x\}}{2k_1}\right]^{-1}, \tag{32}$$

where K_c is given by (21),

$$c_a^2 = \tilde{c}_1^2\left[1-\beta_1(2x)+\int_x^\infty \beta_1(x+z)\,\beta_0(z+x)\,dz - \iint_x^\infty \beta_0(x+z_1)\,\beta_0(z_1+z_2)\beta_1(z_2+x)\,dz_1\,dz_2+\ldots\right]^2,$$

$$c_b^2 = \tilde{c}_1^2\left[1-2K_1\left\{\beta_2(2x)-\int_x^\infty \beta_1(x+z)\,\beta_1(z+x)\,dz + \iint_x^\infty \beta_1(x+z_1)\,\beta_0(z_1+z_2)\,\beta_1(z_2+x)\,dz_1\,dz_2-\ldots\right\}\right],$$

$$\beta_n(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(k)}{(K_1-ik)^n}\exp\{ik\zeta\}\,dk \quad (n = 0, 1, \text{ or } 2).$$

Notice that the entire effect of the interaction between the soliton and the tail is to alter $\tilde{c}_1^2$, which (as $t \to \infty$) affects the phase of the soliton; K_c, which represents the oscillatory tail as $t \to \infty$, evolves as if no soliton were present.

The time-dependence of $\beta_n(\zeta)$ requires that each $\beta_n(\zeta) \to 0$ as $t \to \infty$. If db/dk is bounded, one can show that

$$\left|\int_x^\infty \beta_1(\zeta)\,d\zeta\right| < \infty,$$

and all the integrals in c_a^2 and c_b^2 vanish as $t \to \infty$, for all x. Regardless of db/dk, if one follows the soliton, $\beta_n(\zeta)$ has no points of stationary phase, and one still shows that the integrals vanish as $t \to \infty$, with $x/t = 4K_1^2$. In this case of one soliton, therefore, there is no asymptotic effect on the soliton from its interaction with the oscillatory tail, and vice versa.

Benjamin (1971) demonstrated the neutral stability of the solitary wave. From (32) and the decay of the oscillatory tail, it follows that the solitary wave is asymptotically stable with respect to perturbations which affect only the continuous spectrum; i.e. the cumulative effect on the soliton from having interacted with the oscillatory tail vanishes as $t \to \infty$. Zakharov (1971) claimed that this asymptotically vanishing interaction is typical, and that, for N solitons, the continuous spectrum has no asymptotic effect on the solitons. To summarize:

Let $u(x,t)$ denote a solution of the Korteweg–de Vries equation (1) *that evolves from an initial condition, satisfying* (3) *and* (4), *and with exactly one soliton. As $t \to \infty$, the discrete spectrum of the initial condition affects only the soliton, and the continuous spectrum affects only the tail. It appears that this result can be extended to N solitons.*

3. The asymptotic solution

We have seen that although the general solution of (1) consists of a finite number of solitons, an oscillatory tail and their interactions, only the solitons persist as $t \to \infty$. The method of finding an approximate asymptotic solution is therefore much simpler than the method outlined above. In § 3 we consider some approximate methods for finding the number, and the respective amplitudes, of solitons that emerge from arbitrary initial data. All these methods depend on the close relation between the solitons and the negative eigenvalues of (9).

We consider first the number of solitons N that emerge from an arbitrary initial disturbance $u(x,0)$. Zabusky (1968) claimed that

$$\int_{-\infty}^{\infty} u(x,0)\,dx > 0$$

implies $N \geqslant 1$. An upper bound on N is given by the following extension of the inequality of Bargmann (1952):

$$N \leqslant 1+\int_{-\infty}^{\infty} |x|\, q(x)\,dx, \tag{6a}$$

where

$$q(x) = \begin{cases} u(x,0), & u(x,0) \geqslant 0, \\ 0, & u(x,0) < 0. \end{cases} \tag{6b}$$

The inequality is trivially satisfied if $N \leqslant 1$, so we need consider only $N > 1$. Replace $u(x,0)$ in (9) by $q(x)$, and let N' denote the number of discrete eigenvalues for $q(x)$. Then $N' \geqslant N$. Let $\phi(x)$ satisfy

$$\phi''(x)+q(x)\,\phi(x) = 0. \tag{33}$$

$\phi(x)$ can be chosen to have $(N'+1)$ zeros at finite values of x. Let α, β be two successive zeros such that $x = 0 \notin (\alpha,\beta)$. If $N' \geqslant 2$, such a set of zeros exists, $\phi'(\alpha) \neq 0$, and we may define

$$\chi(x) = \phi(x)/\phi'(\alpha).$$

$\chi(x)$ satisfies (33),

$$\chi(\alpha) = \chi(\beta) = 0; \quad \chi'(\alpha) = 1, \quad \chi(x) \geqslant 0 \quad (\alpha \leqslant x \leqslant \beta). \tag{34a–c}$$

Thus,

$$\chi(x) = x-\alpha-\int_{\alpha}^{x} (x-r)\,q(r)\,\chi(r)\,dr. \tag{35}$$

Using (34a, c) and (35) itself,

$$\beta-\alpha = \int_{\alpha}^{\beta} (\beta-r)\,\chi(r)\,q(r)\,dr \leqslant \int_{\alpha}^{\beta} (\beta-r)\,(r-\alpha)\,q(r)\,dr.$$

As figure 3 shows,

$$(\beta-r)\,(r-\alpha) \leqslant |r|\,(\beta-\alpha).$$

Hence,

$$1 \leqslant \int_{\alpha}^{\beta} |r|\,q(r)\,dr. \tag{36}$$

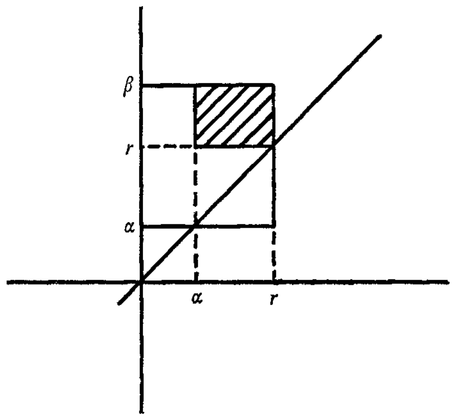

FIGURE 3

$\chi(x)$ has $(N'-1)$ such intervals, excluding the (possible) open interval containing zero. Summing (36) over these $(N'-1)$ intervals,

$$\int_{-\infty}^{\infty} |x|\, q(x)\, dx \geqslant \sum_{\substack{i=1 \\ i\neq i_0}}^{N'} \int_{\alpha_i}^{\beta_i} |x|\, q(x)\, dx \geqslant N'-1 \geqslant N-1,$$

from which (5) follows. Bargmann obtained the strict inequality. If we relax the requirement (3), then (6) cannot be so strengthened. For, if $u(x,0) = Q\delta(x)$, then there exists one eigenvalue if $Q > 0$, and equality is attained in (6).

If $q(x)$ in (6) is bounded and has compact support, as it does in many laboratory experiments, bounds more restrictive than (6) may exist. Considering figure 1, let N, N_1, and N_2 denote the number of discrete eigenvalues when $u(x,0)$, $q_1(x)$, and $q_2(x)$ are each substituted into (9). $N_1 \geqslant N \geqslant N_2$. It is a standard exercise in quantum mechanics textbooks (see Schiff 1949, p. 38) to show that

$$\frac{l_1 Q_1^{\frac{1}{2}}}{\pi} \leqslant N_1 < \frac{l_1 Q_1^{\frac{1}{2}}}{\pi} + 1. \tag{37}$$

Similarly, if

$$\tan(l_2 Q_2^{\frac{1}{2}}) \leqslant \frac{2Q_2^{\frac{1}{2}}}{Q_2 - Q_3}, \quad \frac{l_2 Q_2^{\frac{1}{2}}}{\pi} - 1 < N_2 \leqslant \frac{l_2 Q_2^{\frac{1}{2}}}{\pi}. \tag{38}$$

If

$$\frac{2Q_2^{\frac{1}{2}}}{Q_2 - Q_3} \leqslant \tan(l_2 Q_2^{\frac{1}{2}}), \quad \frac{l_2 Q_2^{\frac{1}{2}}}{\pi} \leqslant N_2 < \frac{l_2 Q_2^{\frac{1}{2}}}{\pi} + 1. \tag{39}$$

Combining these results yields (5).

The amplitude of each of these N waves is $2|\lambda_n|$, where λ_n is the nth negative eigenvalue of $u(x,0)$. The simplest bounds on these amplitudes come directly from (9):

$$0 \leqslant -\lambda_n \leqslant u_0,$$

where $u_0 = \max\{u(x,0)\}$, for any proper eigenvalue corresponding to $u(x,0)$. Let λ_1 denote the most negative eigenvalue; then $|\lambda_1| \to u_0$ as $N \to \infty$.

Let

$$\int_{-\infty}^{\infty} u(x,0)\, dx > 0,$$

and let $u_0 l^2 \ll 1$ where l is a typical scale length. Then only one soliton emerges, and Landau & Lifshitz (1965, p. 156) have shown that

$$|\lambda_1| \doteq \frac{1}{4}\left[\int_{-\infty}^{\infty} u(x,0)\,dx\right]^2. \tag{8}$$

Their work also confirms Zabusky's claim that at least one soliton exists for

$$\int_{-\infty}^{\infty} u(x,0)\,dx > 0,$$

in the case of small initial disturbances.

These bounds are simple to apply, but may not be sufficiently precise in a given situation. More accurate information is found by actually calculating the eigenvalues of (9) with the Rayleigh–Ritz procedure. The largest negative eigenvalue is

$$\lambda_1 = \min_{\psi(x)} \frac{\int_{-\infty}^{\infty} \{(d\psi/dx)^2 - u(x,0)\,\psi^2(x)\}\,dx}{\int_{-\infty}^{\infty} \psi^2(x)\,dx}. \tag{7}$$

The other negative eigenvalues may also be found by this procedure, whose details are omitted. The number of solitons is then found simply by counting the eigenvalues. The asymptotic amplitude of the nth solition is $2|\lambda_n|$.

Observe that, if $u(x,0) \leqslant 0$, no negative eigenvalues exist. Consequently, a sufficient condition for initial data to evolve purely into an oscillatory tail is $u(x,0) \leqslant 0$.

4. The iterated solution

An obvious approach to solving (1) is to try a regular perturbation expansion directly in the differential equation. One seeks a solution of the form

$$u(x,t) = \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + \dots. \tag{40}$$

Collecting powers of ϵ, one obtains a hierarchy of equations:

$$Lu_1 = 0, \quad Lu_2 = -3(u_1^2)_x, \dots Lu_n = -3\sum_{j=1}^{n-1}(u_j u_{n-j})_x, \tag{41}$$

where $Lu = u_t + u_{xxx}$.

Once initial conditions have been prescribed for each equation in (41), the functions $u_n(x,t)$ are well defined. The problem with this approach is that the expansion (40) may not converge. For example, it is known that such an expansion cannot converge to a solitary wave (cf. Stoker 1957, p. 344). In § 4 we show that, in the absence of any solitary waves, initial conditions for (41) can be chosen in such a way that the perturbation expansion (40) is identical (term-by-term) with the Neumann series solution (22). In other words, the perturbation expansion can be made to converge whenever no solitons exist. The identification of these two series is then exploited to obtain information about the solution for large time.

Suppose there are no solitons, so that (22) represents the solution of (1), and let us denote this series by

$$u(x,t) = \sum_{n=1}^{\infty} V_n(x,t), \tag{22a}$$

where $V_n(x,t)$ consists of those terms in (22) in which $b(k)$ is multiplied n times, i.e

$$\left.\begin{aligned} V_1(x,t) &= -2\frac{\partial}{\partial x}B(2x,t),\\ V_2(x,t) &= 4B^2(2x,t),\\ V_3(x,t) &= 4B(2x,t)\int_x^{\infty} B^2(x+z,t)\,dz\\ &\quad -4\iint_x^{\infty}\frac{\partial B}{\partial x}(x,z_1,t)\,B(z_1+z_2,t)\,B(z_2+x,t)\,dz_1\,dz_2.\end{aligned}\right\} \tag{22b}$$

Choose the initial conditions for (41) so that the two expansions (40) and $(22a)$ are identical for $t = 0$:

$$\epsilon^n u_n(x,0) = V_n(x,0). \tag{42}$$

Then it will be shown by induction that V_n satisfies the same differential equation as $\epsilon^n u_n$, so that the two series are identical for all time.

The series $(22a)$, which is like a power series in $b(k)$, satisfies the differential equation (1), regardless of the choice of $b(k)$. The 'powers' of $b(k)$ are linearly independent, so each 'power' of $b(k)$ in (1) must vanish identically. Substituting $(22a)$ into (1), one obtains an equation for each V_n:

$$LV_1 = 0, \quad LV_2 = -6V_1(V_1)_x = -3(V_1^2)_x, \ldots LV_n = -3\sum_{j=1}^{n-1}(V_jV_{n-j})_x. \tag{43}$$

(Using (19) and $(22b)$, these equations can be verified algebraically.) Equation (41) differs from (43) only in the right-hand sides. But the equation for V_1 is homogeneous, as is the corresponding equation for u_1. Hence, V_1 and ϵu_1 are equal for $t = 0$, satisfy the same differential equation, and must be identical for all time. But then the right-hand sides of the equation for V_2 and $\epsilon^2 u_2$ are identical, so V_2 and $\epsilon^2 u_2$ must be identical.

Assume now that V_j and $\epsilon^j u_j$ are identical for all $j \leqslant n-1$. Then the right-hand sides of the equations for V_n and $\epsilon^n u_n$ are identical. Hence V_n and $\epsilon^n u_n$ are equal for $t = 0$, satisfy the same differential equation, and are identical for all time. It follows that the series (40) and (22) are identical, term-by-term.

If solitons are present, (22) does not represent the solution of (1), and the procedure breaks down. To summarize: *when* (3) *and* (4) *are satisfied, the iterated solution* (40) *of the differential equation* (1) *can be made to converge if and only if no solitons exist.*

The correspondence between the two series has some interesting consequences. Every term in (40) satisfies a set of equations of the form

$$f_t + f_{xxx} = \rho(x,t) \quad f(x,0) = \phi(x), \tag{44}$$

where ρ and ϕ are continuous and vanish as $|x| \to \infty$. The solution of (44) is

$$f(x,t) = \int_{-\infty}^{\infty}\phi(\xi)\,h(x-\xi,t)\,d\xi + \int_{-\infty}^{\infty}\int_0^t \rho(\xi,\tau)\,h(x-\xi,t-\tau)\,d\tau\,d\xi, \tag{45}$$

where $$h(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{ik^3t+ikr\}\,dk = (3t)^{-\frac{1}{3}}\,\mathrm{Ai}\left(\frac{r}{(3t)^{\frac{1}{3}}}\right),$$

and Ai (z) is an Airy function. For large time, these integrals evolve into slowly varying wave trains, whose group velocity is given by

$$x/t = -3k^2. \tag{24}$$

Hence, for large time, $f(x,t)$ is a slowly varying wave train in which the wave-number k dominates near the location defined by (24). But every term in the expansion (40) for $u(x,t)$ can be written in this form. Thus, the group velocity of every component of u, and therefore u itself, is defined by (24). This information was used in § 2.3 to determine the algebraic decay rate of the solution corresponding to the continuous spectrum.

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REFERENCES

AGRANOVICH, Z. S. & MARCHENKO, V. A. 1963 *The Inverse Problem of Scattering Theory* (trans. B. D. Seckler). Gordon & Breach.

BARGMANN, V. 1952 On the number of bound states in a central field of force. *Proc. N.A.S.*, **38**, 961.

BENJAMIN, T. B. 1971 The stability of solitary waves. *University of Essex Fluid Mech. Res. Inst. Rep.* no. 18.

BENNEY, D. J. & LUKE, J. C. 1964 On the interaction of permanent waves of finite amplitude. *J. Math. & Phys.* **43**, 309.

BEREZIN, Y. A. & KARPMAN, V. I. 1964 Theory of nonstationary finite-amplitude waves in a low-density plasma. *Sov. Phys. J.E.T.P.*, **19**, 1265.

BEREZIN, Y. A. & KARPMAN, V. I. 1967 Nonlinear evolution of disturbances in plasmas and other dispersive media. *Sov. Phys. J.E.T.P.*, **24**, 1049.

BONA, J. & SMITH, R. 1973 The Korteweg–de Vries equation in an unbounded domain. (To be published.)

BYATT-SMITH J. G. B. 1971 An integral equation for unsteady surface waves. *J. Fluid Mech.* **49**, 625.

FADDEEV, L. D. 1958 On the relation between the *S*-matrix and potential for the one-dimensional Schrödinger operator. *Sov. Phys. Doklady*, **3**, 747.

GARDNER, C. S., GREENE, S. M., KRUSKAL, M. P. & MIURA, R. M. 1967 Method for solving the Korteweg–de Vries equation. *Phys. Rev. Lett.* **19**, 1095.

GELFAND, I. M. & LEVITAN, B. M. 1955 On the determination of a differential equation from its spectral function. *Am. Math. Soc. Translations* **1**, (2), 253.

HAMMACK, J. L. & SEGUR, H. 1973 The Korteweg–de Vries equation and water waves. Part 2. (To be published.)

KARPMAN, V. I. 1967 On the structure of two-dimensional flow around bodies in dispersive media. *Akad. Nauk. SSSR*, **52**, 1657.

KARPMAN, V. I. & SOKOLOV, V. P. 1968 On solitons and eigenvalues of the Schroedinger equation. *Akad. Nauk. SSSR*, **54**, 1568.

KAY, I. & MOSES, H. E. 1956 Reflectionless transmission through dielectrics and scattering potentials. *J. Appl. Phys.* **27**, 1503.

KORTEWEG, D. J. & DE VRIES, G. 1895 On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.* **39**, (5), 422.

LANDAU, L. D. & LIFSHITZ, E. M. 1965 *Quantum Mechanics* (trans. J. B. Sykev and J. S. Bell). Pergamon.

LAX, P. D. 1968 Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* **21**, 467.

SCHIFF, L. I. 1949 *Quantum Mechanics*. McGraw-Hill.

STOKER, J. J. 1957 *Water Waves*. Interscience.

ZABUSKY, N. J. 1968 Solitons and bound-states of the time-independent Schrödinger equation. *Phys. Rev.* **168**, 124.

ZAKHAROV, V. E. 1971 Kinetic equations for solitons. *Sov. Phys. J.E.T.P.* **33**, 538.

see Ablowitz & Newell 1973 J. Mathematical Physics 14 1277-1284
for more precise description of the tail.